SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.14.1.03
Vol.14 (27), No.1, (2018), 25–33

THE IDEAL-BASED TRIPLE ZERO GRAPH OF A COMMUTATIVE RING

ECE YETKIN CELIKEL

ABSTRACT. Let R be a commutative ring with $1 \neq 0$ and I a proper ideal of R. The primary focus of this paper is a generalization of the triple zero graph which is called the ideal-based triple zero graph. Let $TZ_I(R) = \{x \in R : xyz \in I$ for some $y,z \in R$ such that $xy, yz, xz \notin I\}$. In this paper, we introduce the ideal based triple zero graph of R with respect to I denoted by $TZ\Gamma_I(R)$ with vertices $TZ_I(R)$. Two distinct vertices x and y are adjacent if and only if $xy \notin I$ and $xyz \in I$ for some $z \in R$ such that $yz, xz \notin I$.

1. INTRODUCTION

Throughout this paper all rings are commutative with $1 \neq 0$. The concept of zero-divisor graph of R was defined by I. Beck [11]. According to Beck's definition, every graph contained a star subgraph where 0 was adjacent to every other vertex. Then the zero-divisor graph has been studied by D.D. Anderson, M. Naseer [1] with the following definition: The set of vertices of this graph is R and distinct vertices $x, y \in R$ are adjacent if and only if $xy = 0$. In the late 1990s, D. F. Anderson and P. S. Livingston [3] modifed Beck's definition to the following: Let the vertices of the graph be $Z(R)^* = Z(R) \backslash \{0\}$, where $Z(R) = \{x \in R : xy = 0$ for some $y \in R \backslash \{0\}\}$ is the set of zero-divisors of R, and distinct vertices x and y are adjacent if and only if $xy = 0$. By this study, they focus on relations between ring-theoretic properties and graph-theoretic properties. Many authors have studied the zero-divisor graph in [2], [3], [4], [5]. In 2001, S. P. Redmond gave the following definition in [14], [15]. Let R be a commutative ring with nonzero identity and I an ideal of R. Define $\Gamma_I(R)$ to be the graph on vertices $\{x \in R \backslash I : xy \in I$ for some $y \in R \backslash I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. This is called the ideal-based zero-divisor graph of R with respect to the ideal I. Ideal-based zero divisor graphs are studied by many authors [6], [7]. Recently, in [16], the triple zero graph of a commutative ring $TZ\Gamma(R)$ has been investigated.

2010 *Mathematics Subject Classification.* 05C38, 15A15, 15A18.

Key words and phrases. Zero-divisor graph, ideal-based zero-divisor graph, triple zero graph, ideal-based triple zero graph.

Let $TZ(R) = \{a \in Z(R)$: there exists $b, c \in R$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0\}$. The triple zero graph of R is an undirected graph $TZ\Gamma(R)$ with vertices $TZ(R)$ and two distinct vertices a and b are adjacent if and only if $ab \neq 0$ and there is an element $c \in R$ such that $abc = 0$, and $bc, ac \neq 0$. Let I be a proper ideal of a commutative ring R. In this paper, we introduce the ideal based triple zero graph of a commutative ring R respect to the ideal I denoted by $TZ\Gamma_I(R)$. This graph is a very natural generalization of the triple zero graph in [16].

For the sake of completeness, we state in this section some definitions from graph theory which will be used in the sequel. Let G be a (undirected) graph. The order of G, denoted by $|G|$, is equal to the cardinality of the vertex set. At times, we will let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G. A graph isomorphism is a bijection between the vertex sets which preserves edges. A graph H is a subgraph of G, denoted $H \subseteq G$, if the $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It is well known that G is connected if there is a path between any two distinct vertices. For vertices a and b of G, we say that the distance between a and b, $d(a,b)$, is the length of a shortest path from a to b. If there is no path between a and b, then $d(a,b) = \infty$, and $d(a,a) = 0$. A graph G is said to be totally disconnected if it has no edges. The diameter of G is defined by $diam(G) = \sup\{d(a,b) : a$ and b are vertices of $G\}$. A cycle in a graph is a path that begins and ends at the same vertex. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G. If G contains no cycles, then $gr(G) = \infty$. A cycle of length three is commonly called a triangle, a cycle of length four is a square. A graph is said to be triangle-free graph if no three vertices form a triangle of edges. A graph G is complete if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K^n. A complete bipartite graph is a graph G which may be partioned into two disjoint non-empty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. We denote the complete bipartite graph by $K^{m,n}$, where $|A| = m$ and $|B| = n$. If one of the vertex sets is a singleton, then we call G a star graph because of its resemblence to a star shape. A star graph is clearly a $K^{1,n}$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo n, respectively. For an ideal I of R, the radical of I is defined to be $\sqrt{I} = \{a \in R : a^n \in I$ for some $n \in \mathbb{N}\}$. We will denote the ideal of nilpotent elements of a ring R by $Nil(R) = \sqrt{0}$. We say that ring R is reduced if $Nil(R) = 0$. For general background and terminology, the reader may consult [12], [8], [13].

2. Properties of the ideal-based triple zero graph

Let R be a commutative ring with $1 \neq 0$. In this section, we generalize the notion of the triple zero graph to the triple zero graph of R based on a nonzero ideal I of R. To avoid trivialities, we always assume $I \neq R$. We investigate basic properties of the triple zero graph of R with respect to I.

Definition 2.1. Let I be a proper ideal of a commutative ring R. We define an undirected graph $TZ\Gamma_I(R)$ with vertices $TZ_I(R) = \{a \in R :$ there exists $b, c \in R$ such that $abc \in I$, and $ab, bc, ac \notin I\}$, where distinct vertices a and b are adjacent if and only if there is some $c \in R$ such that $abc \in I$, and $ab, bc, ac \notin I$.

Example 2.2.

(1) Let $R = \mathbb{Z}_2[X]/(X^4)$ and $I = (X^3)$. Then $TZ\Gamma_I(R) \cong K^2$. Indeed, $X - X + X^2$ as $X \cdot X \cdot (X + X^2) = X^3 \in I$, but $X \cdot X \notin I$ and $X \cdot (X + X^2) \notin I$.

(2) Let $R = \mathbb{Z}_p[X]/(X^4)$ and $I = (X^3)$. Then the vertices of $TZ\Gamma_I(R)$ are in the form of $aX + bX^2$, where $a \in \{1, \ldots, p-1\}$ and $b \in \{0, \ldots, p-1\}$; so the number of vertices is $p(p-1)$. Since $(aX)(bX)(cX) \in I$, and $(aX)(bX)$, $(bX)(cX)$, $(aX)(cX) \notin I$, for all $a, b, c \in \{1, \ldots, p-1\}$, all vertices in the form of kX, where $k \in \{1, \ldots, p-1\}$ are adjacent. Let $a, b, d \in \{1, \ldots, p-1\}$ and $c, e \in \{0, \ldots, p-1\}$. Also observe that $(aX)(bX + cX^2)(dX + eX^2) \in I$, and $(aX)(bX + cX^2) \notin I$, $(aX)(dX + eX^2) \notin I$, $(bX + cX^2)(dX + eX^2) \notin I$. Hence $aX - bX + cX^2 - dX + eX^2$. Thus $TZ\Gamma_I(R)$ is complete; i.e., $TZ\Gamma_I(R) \cong K^{p^2 - p}$.

(3) Let $R = \mathbb{Z}_{p^n}$ and $I = (p^3)$, where p is a prime integer and $n \geq 3$. Then $TZ\Gamma_I(R) \cong K^{\phi(p^{n-1})}$, where ϕ is the Euler's function. Indeed, the triple zero graph of $\mathbb{Z}_{p^n}$ with respect to its ideal (p^3) is a complete graph with the vertices of the form kp, where $(k, p) = 1$ and $kp < p^n$. So $\left|V(TZ\Gamma_{(p^3)}(\mathbb{Z}_{p^n})\right| = \left|k : 0 < k < p^{n-1}\right| = \phi(p^{n-1})$.

Let n be an even integer. Then the following example shows that there is a ring R with a nonzero ideal I such that $TZ\Gamma_I(R) \cong K^n$.

Example 2.3. Consider $R = \mathbb{Z}_n \times \mathbb{Z}_2[X]/\left(X^3\right)$ and $I = \mathbb{Z}_n \times 0$. Then the vertices of $TZ\Gamma_I(R)$ are of the type (a, X) and $(b, X + X^2)$, where $a, b \in \mathbb{Z}_n$, and observe that this graph is complete. Thus $TZ\Gamma_I(R) \cong K^{2n}$.

Example 2.4. Let $R = \mathbb{Z}_{210}$ and consider the ideal $I = (105) = \{0, 105\}$ of R. Since $3 \cdot 5 \cdot 7 = 105 \in I$ and neither $3 \cdot 5$, $3 \cdot 7$ nor $5 \cdot 7$ is in I, the vertices $3, 5, 7$ of $TZ\Gamma_I(R)$ are adjacent. Also the other adjacent vertices (i.e., triangles) are as shown in Figure 1.

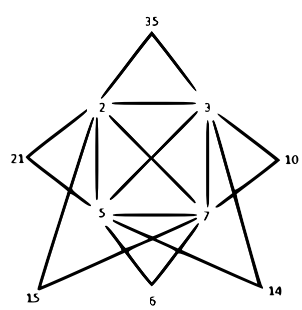

FIGURE 1. $TZ\Gamma_{(105)}(\mathbb{Z}_{210})$

On the other hand, the reader can observe that the vertices of the ideal-based zero-divisor graph $\Gamma_I(R)$ are $V(\Gamma_I(R)) = \{3,5,6,7,9,10,...,205\}$. However, $TZ\Gamma_I(R)$ is not a subgraph of $\Gamma_I(R)$ as 2 is a vertex of $TZ\Gamma_I(R)$ while 2 is not a vertex of $\Gamma_I(R)$.

Example 2.5. Let $R_n = \mathbb{Z}_2 \times \mathbb{Z}_{p^n}$ and $I = \mathbb{Z}_2 \times \{0\}$, where p is a prime number and n is a positive integer. It can be easily seen that $TZ\Gamma_I(R_1)$ and $TZ\Gamma_I(R_2)$ are the empty graph for all prime numbers p. Suppose that $p = 2$ and $n = 3$. Then $gr(TZ\Gamma_I(R_3)) = 3$. Indeed, we obtain a triangle $(0,2)-(0,6)-(1,6)-(0,2)$. More generally, for $n \geq 3$, we obtain a triangle $(0,p)-(0,kp)-(1,kp)-(0,p)$, where k is a integer distinct from p with $1 < k < p^{n-1}$. Thus $diam(TZ\Gamma_I(R_n)) = 2$ and $gr(TZ\Gamma_I(R_n)) = 3$ for $n \geq 3$.

Theorem 2.6. *Let I be a proper ideal of R. If $TZ_I(R)$ is an ideal of R, then $TZ_I(R)$ is a prime ideal of R.*

Proof. Suppose that $ab \in TZ_I(R)$ for some $a,b \in R$. Then there exists $x,y \in R$ such that $(ab)xy \in I$ and $abx, aby, xy \notin I$. There are two cases:

Case I. Assume that $axy \in I$. Since $ax, ay, xy \notin I$, we have $a \in TZ_I(R)$.

Case II. Let $axy \notin I$. Suppose that $bxy \notin I$. Since $abxy \in I$, $axy, bxy, ab \notin I$, we conclude that $b \in TZ_I(R)$. So assume that $bxy \in I$. Since $xy, bx, by \notin I$, we obtain $b \in TZ_I(R)$. Thus $TZ_I(R)$ is a prime ideal of R. □

As a generalization of prime ideals, A. Badawi introduced the concept of 2-absorbing ideal of a commutative ring in [9]. A proper ideal I of R is said to be 2-absorbing if $a,b,c \in R$ with $abc \in I$ implies that either $ab \in I$ or $bc \in I$ or $ac \in I$. As mentioned in the introduction by $\Gamma_I(R)$ we mean the ideal-based zero-divisor graph of R on vertices $\{x \in R \backslash I : xy \in I$ for some $y \in R \backslash I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.

Proposition 2.7. *Let I be an ideal of R. Then the following properties hold:*

(1) If $I = \{0\}$, then $TZ\Gamma_I(R) = TZ\Gamma(R)$.
(2) $TZ(R/I) = TZ_I(R)/I$.
(3) If $\Gamma_I(R) = \emptyset$, then $TZ\Gamma_I(R) = \emptyset$.
(4) $TZ\Gamma_I(R) = \emptyset$ if and only if I is a 2-absorbing ideal of R.

Proof. The proofs of (1), (2), and (3) are clear from definitions.

(4) Suppose that $TZ\Gamma_I(R) = \emptyset$ and $a,b,c \in R$ with $abc \in I$. Since there is no triple zero element of I, we conclude that either $ab \in I$ or $bc \in I$ or $ac \in I$. Thus I is a 2-absorbing ideal of R. Conversely suppose that I is a 2-absorbing ideal of R and $a \in TZ_I(R)$. Then $abc \in I$, $ab, bc, ac \notin I$ for some $b,c \in R$, a contradiction. □

Note that if $I = P_1 \cap P_2$, where P_1 and P_2 are prime ideals of R, then $TZ\Gamma_I(R)$ is empty. Indeed, the intersection of two prime ideals is a 2-absorbing ideal of R by [9]; so this graph is empty by Proposition 2.7(4).

Theorem 2.8. *Let I be a nonzero ideal of R. If P_1, P_2, and P_3 are prime ideals of R, and $I = P_1 \cap P_2 \cap P_3$, then $TZ\Gamma_I(R)$ is a complete tripartite graph.*

Proof. Suppose that $a,b,c \in R$ such that $abc \in I$ with $ab,bc,ac \notin I$. Then $abc \in P_i$ for all $i = 1,2,3$. Since every P_i is prime, we obtain that $a \in P_i$ or $b \in P_i$ or $c \in P_i$ for each $i = 1,2,3$. As $ab,bc,ac \notin I$, without loss generality, we assume that $a \in P_1 \setminus (P_2 \cup P_3)$, $b \in P_2 \setminus (P_1 \cup P_3)$, and $c \in P_3 \setminus (P_1 \cup P_2)$. Therefore $TZ\Gamma_I(R)$ is a complete tripartite graph with parts $P_1 \setminus (P_2 \cup P_3)$, $P_2 \setminus (P_1 \cup P_3)$, and $P_3 \setminus (P_1 \cup P_2)$. □

Theorem 2.9. *Let R is a quasi-local ring with maximal ideal M such that $M^2 = \{0\}$. Then $TZ\Gamma_I(R) = \emptyset$ for all ideals I of R.*

Proof. Suppose that R is a quasi-local ring with maximal ideal M such that $M^2 = \{0\}$. From Corollary 3.3 [10], all ideals of R are 2-absorbing. Thus the ideal-based triple zero graph is empty for all ideals I of R by Proposition 2.7(4). □

Proposition 2.10. *Let I be an ideal of a ring R. If $TZ\Gamma_I(R)$ has exactly one vertex, then $I = \{0\}$.*

Proof. Suppose that $a \in TZ_I(R)$ is the only vertex of the graph $TZ\Gamma_I(R)$. Then $a^3 \in I$, but $a^2 \notin I$. Let $i \in I$. Since $a \cdot a \cdot (a+i) = a^3 + a^2 i \in I$, $a \cdot (a+i) = a^2 + ai \notin I$ and $a^2 \notin I$, we conclude that $a - (a+i)$. Hence $a = a+i$ as a is the only vertex of the graph. Thus $i = 0$, and so $I = \{0\}$. □

Theorem 2.11. *Let R_1 and R_2 be commutative rings with identity, $R = R_1 \times R_2$, and I be an ideal of R_1. Then the following hold.*

(1) $TZ\Gamma_I(R_1) = \emptyset$ if and only if $TZ\Gamma_{I\times R_2}(R) = \emptyset$.
(2) $TZ\Gamma_I(R_1)$ is complete if and only if $TZ\Gamma_{I\times R_2}(R)$ is complete.

Proof. Observe that if a is adjacent to b in $TZ\Gamma_I(R_1)$, then (a,r) is adjacent to (b,s) in $TZ\Gamma_{I\times R_2}(R)$ for all $r,s \in R_2$; and the converse is also true. Thus the results are clear. □

Theorem 2.12. *Let R_1 and R_2 be commutative rings, I_1, I_2 ideals of R_1, R_2, respectively. Let $R = R_1 \times R_2$ and $I = I_1 \times I_2$. If $R_1 = I_1$ and $diam(TZ\Gamma_{I_2}(R_2)) > 0$, then $diam(TZ\Gamma_I(R)) = diam(TZ\Gamma_{I_2}(R_2))$.*

Proof. Let $diam(TZ\Gamma_I(R)) = n$ and $diam(TZ\Gamma_{I_2}(R_2)) = m$. First suppose that $n > m$. Then there are $(a_0,b_0),(a_1 b_1),...,(a_n,b_n) \in TZ_I(R)$ such that $(a_0,b_0) - (a_1 b_1) - \cdots - (a_n,b_n)$ is a minimal path. Hence we get $b_0 - b_1 - \cdots - b_n$ which is a path in $TZ\Gamma_{I_2}(R_2)$. Since $diam(TZ\Gamma_{I_2}(R_2)) < n$, this path can not be a minimal path. Here we have the following cases:

Case I. Assume that $b_i - b_j$ for some $i, j \in \{0,...,n\}$ with $i+1 < j$. Hence we obtain $(a_i,b_i) - (a_j,b_j)$ which contradicts $(a_0,b_0) - (a_1,b_1) - \cdots - (a_n,b_n)$ being a minimal path.

Case II. Assume that $b_0 - c - b_n$ for some $c \in TZ_{I_2}(R_2)$ which is distinct from b_i for all $i = 0, ..., n$. Hence we obtain $(a_0, b_0) - (0, c) - (a_n, b_n)$, a contradiction again. Thus $n \leq m$.

Now suppose $n < m$. Then there is a minimal path $b_0 - b_1 - \cdots - b_m$ for some $b_0, b_1, ..., b_m \in TZ_{I_2}(R_2)$. Then we obtain a minimal path of length m : $(r_0, b_0) - (r_1, b_1) - \cdots - (r_m, b_m)$ in $TZ\Gamma_I(R)$ for all $r_0, r_1, ..., r_m \in R_1$ as $R_1 = I_1$, which is a contradiction. Thus $n \geq m$. Consequently, we have $n = m$. □

Theorem 2.13. *Let R_1, R_2, R_3 be commutative reduced rings (not necessarily with identity), $R = R_1 \times R_2 \times R_3$, and $diam(TZ\Gamma(R_1)) = diam(TZ\Gamma(R_2))$ $= diam(TZ\Gamma(R_3)) = 0$. Then $diam(TZ\Gamma(R)) = 0$ if and only if $R_1 = \{0\}$ or $R_2 = \{0\}$ or $R_3 = \{0\}$.*

Proof. Suppose on the contrary that $R_1 \neq \{0\}$, $R_2 \neq \{0\}$, and $R_3 \neq \{0\}$. Hence there are some elements $a \in R_1 \backslash \{0\}$, $b \in R_2 \backslash \{0\}$, $c \in R_3 \backslash \{0\}$. Since $(a, b, 0) - (0, b, c) - (a, 0, c)$ is a path of length 2, $diam(TZ\Gamma(R)) \neq 0$. Conversely, if $R_1 = R_2 = \{0\}$, then $R \cong R_3$, so $diam(TZ\Gamma(R)) = 0$. Without loss of generality, if $R_1 = \{0\}$ and $R_2, R_3 \neq \{0\}$, then $R \cong R_2 \times R_3$, and it is easy to see that there is no triple zero element of $R_2 \times R_3$. Thus $diam(TZ\Gamma(R)) = 0$. □

Theorem 2.14. *Let I be an ideal of a ring R, and let $a, b \in R \backslash I$. Then the following hold.*

(1) If $a + I$ and $b + I$ are adjacent in $TZ\Gamma(R/I)$, then a and b are adjacent in $TZ\Gamma_I(R)$.
(2) If a and b are adjacent in $TZ\Gamma_I(R)$ and $a + I \neq b + I$, then $a + I$ and $b + I$ are adjacent in $TZ\Gamma(R/I)$.
(3) If a is adjacent to b in $TZ\Gamma_I(R)$ and $a + I = b + I$, then $a^2, b^2 \notin I$, but $a^2c \in I$ and $b^2d \in I$ for some $c, d \in R$.
(4) If $TZ\Gamma_I(R) = \emptyset$, then $TZ\Gamma(R/I) = \emptyset$.

Proof. (1) Suppose that $a + I$ and $b + I$ are adjacent in $TZ\Gamma(R/I)$. Hence there is some $c + I \in R/I$ such that $(a + I)(b + I)(c + I) = abc + I = 0 + I$, $(a + I)(b + I) = ab + I \neq 0 + I$, $(a + I)(c + I) = ac + I \neq 0 + I$, and $(b + I)(c + I) = bc + I \neq 0 + I$. This means that $abc \in I$, but neither $ab \in I$, $bc \in I$, nor $ac \in I$. Thus a is adjacent to b in $TZ\Gamma_I(R)$.

(2) Suppose that $a - b$ in $TZ\Gamma_I(R)$ and $a + I \neq b + I$. Then $abc \in I$, $ab, bc, ac \notin I$ for some $c \in R$. Hence $(a + I)(b + I)(c + I) = abc + I = 0 + I$, $(a + I)(b + I) = ab + I \neq 0 + I$, $(a + I)(c + I) = ac + I \neq 0 + I$, and $(b + I)(c + I) = ab + I \neq 0 + I$. Therefore $a + I$ is adjacent to $b + I$ in $TZ\Gamma(R/I)$.

(3), (4) are clear. □

Theorem 2.15. *Let I be an ideal of a ring R. Then $gr(TZ\Gamma_I(R)) \leq gr(TZ\Gamma(R/I))$.*

Proof. Suppose that $gr(TZ\Gamma(R/I)) = \infty$. Then we are done. So suppose $gr(TZ\Gamma_I(R)) = n$. If $a_1 + I - a_2 + I - \cdots - a_n + I - a_1 + I$ is a cycle in $TZ\Gamma(R/I)$

through n distinct vertices, then $a_1 - a_2 - \cdots - a_n - a_1$ is a cycle in $TZ\Gamma_I(R)$ of length n by Theorem 2.14(1). Thus $gr(TZ\Gamma_I(R)) \leq n$. □

Theorem 2.16. *Let R be a commutative ring with nonzero identity and I an ideal of R. If $TZ\Gamma(R/I) \cong K^1$, then $TZ\Gamma_I(R) \cong K^{|I|}$.*

Proof. Suppose that $a+I$ is the vertex of $TZ\Gamma(R/I)$. Hence we have $a^3 \in I$ and $a^2 \notin I$. Observe that $(a+i_1)(a+i_2)(a+i_3) \in I$, but $(a+i_j)(a+i_k) \notin I$ for all $j,k = 1,2,3$. Thus $TZ\Gamma_I(R)$ is a complete graph on $|I|$ vertices. □

Corollary 2.17. *Let I be a nonzero ideal of a ring R. If $TZ\Gamma(R/I)$ has only one vertex, then* $gr(TZ\Gamma_I(R)) = \begin{cases} 3 & \text{if } |I| \geq 3 \\ \infty & \text{if } |I| = 2 \end{cases}$.

Note that there is a relationship between $TZ\Gamma(R/I)$ and $TZ\Gamma_I(R)$ which is similar to the relationship between $\Gamma(R/I)$ and $\Gamma_I(R)$ given in [14]. Let I be an ideal of a ring R. Then we can construct the graph $TZ\Gamma_I(R)$. Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq R$ be a set of coset representatives of the vertices of $TZ\Gamma(R/I)$. For each $i \in I$, define a graph G_i with vertices $\{a_\lambda + i : \lambda \in \Lambda\}$, where $a_\lambda + i$ and $a_\alpha + i$ are adjacent in G_i if and only if $a_\lambda + I$ and $a_\alpha + I$ are adjacent in $TZ\Gamma(R/I)$. Now we can define a graph G with vertices $V(G) = \cup_{i\in I} G_i$ and which satisfies the following property for edges: G contains all edges which are in G_i for all $i \in I$ and for $\lambda \neq \alpha \in \Lambda$, $a_\lambda + i$ is adjacent to $a_\alpha + i$ in G_i if and only if $a_\lambda + I$ is adjacent to $a_\alpha + I$ in $TZ\Gamma(R/I)$. For $i \neq j \in I$, $a_\lambda + i$ is adjacent to $a_\alpha + j$ if and only if $a_\lambda^2, a_\alpha^2 \notin I$, and there exist $c, d \in R$ such that $a_\lambda^2 c \in I$ and $a_\alpha^2 d \in I$.

Let R and S be commutative rings and I, J ideals of them, respectively. The following example shows that $TZ\Gamma(R/I) \cong TZ\Gamma(S/J)$ does not imply that $TZ\Gamma_I(R) = TZ\Gamma_J(S)$.

Example 2.18. Consider the ideal $I = \mathbb{Z}_2 \times 0$ of $R = \mathbb{Z}_2 \times \mathbb{Z}_8$. Then observe that $R/I \cong \mathbb{Z}/(8)$; so $TZ\Gamma(R/I) \cong TZ\Gamma(\mathbb{Z}/(8))$. However, $TZ\Gamma_I(R) \cong K^4$ as illustrated in Figure 2, but $TZ\Gamma_{(8)}(\mathbb{Z})$ is not. Indeed $TZ\Gamma_{(8)}(\mathbb{Z})$ has infinitely many vertices: 2, $2k$, where k is an odd integer.

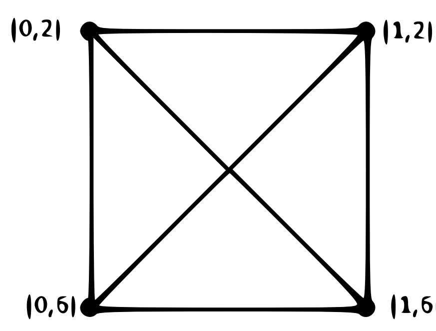

FIGURE 2. $TZ\Gamma_{\mathbb{Z}_2\times 0}\mathbb{Z}_2 \times \mathbb{Z}_8$

Proposition 2.19. *If a and b are distinct adjacent vertices in $TZ\Gamma_I(R)$, then all distinct elements of $a+I$ and $b+I$ are adjacent in $TZ\Gamma_I(R)$. If $a^2 \in I$, then none of the distinct elements of $a+I$ are adjacent in $TZ\Gamma_I(R)$.*

Proof. Suppose that $a-b$ in $TZ\Gamma_I(R)$. Then $abc \in I$, and $ab, bc, ac \notin I$ for some $c \in R$. Then $(a+i_1)(b+i_2)(c+i_3) \in I$, and $(a+i_1)(b+i_2)$, $(a+i_1)(c+i_3)$, $(b+i_2)(c+i_3) \notin I$ for all $i_1, i_2, i_3 \in I$. Thus all distinct elements of $a+I$ and $b+I$ are adjacent in $TZ\Gamma_I(R)$. Now suppose that $a^2 \in I$. Since $(a+i)\cdot(a+j) \in I$ for all $i, j \in I$, the result is clear. □

Let I be an ideal of R. By using the similar idea in [14], we can define a connected column of $TZ\Gamma_I(R)$ as follows:

Definition 2.20. We call the subset $a_\lambda + I$ a column of $TZ\Gamma_I(R)$. If $a_\lambda^2 \notin I$, but $a_\lambda^2 c \in I$ for some $c \in R$, then we call $a_\lambda + I$ a connected column of $TZ\Gamma_I(R)$.

Theorem 2.21. *Let I be an ideal which has two elements of a ring R. If $TZ\Gamma(R/I)$ has at least two vertices and $TZ\Gamma_I(R)$ has at least one connected column, then $gr(TZ\Gamma_I(R)) = 3$.*

Proof. Suppose that $a+I$ is a connected column of $TZ\Gamma_I(R)$, and $a+I$, $b+I$ are adjacent vertices of $TZ\Gamma_I(R)$. Hence $abc \in I$, and $ab, bc, ac \notin I$ for some $c \in R$. Since $a+I$ is a connected column, $a^2 \notin I$ and $a^2 d \in I$ for some $d \in R$. Let $I = \{0, i\}$. Then we obtain a cycle $b - a - (a+i) - b$ of length 3 in $TZ\Gamma_I(R)$. Thus $gr(TZ\Gamma_I(R)) = 3$. □

Theorem 2.22. *Let I be an ideal of a ring R. If $|I| \geq 3$ and $TZ\Gamma_I(R)$ contains a connected column, then $gr(TZ\Gamma_I(R)) = 3$.*

Proof. Suppose that $a+I$ be a connected column of $TZ\Gamma_I(R)$. Then $a^2 \notin I$ and $a^2 c \in I$ for some $c \in R$. Since $|I| \geq 3$, there are $i, j \in I \backslash \{0\}$. Hence $a - (a+i) - (a+j) - a$ is a cycle of length 3 in $TZ\Gamma_I(R)$. Thus $gr(TZ\Gamma_I(R)) = 3$. □

Recall from [10] that an ideal I of a ring R is called weakly 2-absorbing ideal if whenever $a, b, c \in R$ with $0 \neq abc \in I$, then either $ab \in I$, $bc \in I$, or $ac \in I$.

Lemma 2.23. [10] *Let I be a weakly 2-absorbing ideal of R. If I is not 2-absorbing ideal of R, then $I^3 = 0$.*

Let $S_I(R) = \{x \in R : x^3 = 0\}$. Note that if I is a weakly 2-absorbing ideal which is not 2-absorbing, then $I \subseteq S_I(R) \subseteq Nil(R)$.

Theorem 2.24. *Let I be a weakly 2-absorbing ideal of a commutative ring R. Then $V(TZ\Gamma_I(R)) \subseteq \{a \in R : abc = 0$ for some $b, c \in R$ such that $ab, bc, ac \notin I\}$. Additionally, if R is an integral domain, then $TZ\Gamma_I(R) = \emptyset$.*

Proof. Suppose that I is a weakly 2-absorbing ideal of R, and
$a \in V(TZ\Gamma_I(R))$. Hence there are $b, c \in R$ with $abc \in I$, but $ab, bc, ac \notin I$. If $abc \neq$

0, then we conclude that either $ab \in I$, $bc \in I$, or $ac \in I$, a contradiction. Thus $abc = 0$; so we are done. The ”additionally” part is clear. □

The author is grateful to the referees for their comments and valuable suggestions which let to the improvement of the manuscript.

REFERENCES

[1] D.D. Anderson and M. Naseer, *Beck's coloring of commutative rings*, J. Algebra, 159 (1993), 500–514.
[2] D.F. Anderson, M.C. Axtell, and J,A. Stickles,*Zero-divisor graphs in commutative rings*, Commutative Algebra, Springer, New York, (2011), 23–45.
[3] D.F. Anderson and P.S Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217 (1999), 434–447.
[4] D.F. Anderson and A. Badawi, *On the zero-divisor graph of a commutative ring*, Comm. Algebra, 36 (2008), 3073–3092.
[5] D.F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra, 320 (2008), 2706–2719.
[6] D.F Anderson, S. E. Atani, M.S. Kohan, Z.E. Sarvandi, *The ideal-based zero-divisor graph of commutatve chained rings*, Sarajevo J. Math., 10 (22) (2014), 3–12.
[7] S.E. Atani,*An ideal-based zero-divisor graph of a commutative semiring*, Glas. Mat. Ser. III, 44(64) (2009), 14–153.
[8] M. Atiyah, I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, (1969).
[9] A.Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Aust. Math. Soc., 75 (2007), 417–429.
[10] A. Badawi, A.Y. Darani, *On weakly 2-absorbing ideals of commutative rings*, Houston J. Math., 39 (2) (2013), 441–452.
[11] I. Beck, *Coloring of commutative rings*, J. Algebra, 116 (1988), 208–226.
[12] B. Bollaboás, *Graph Theory, An Introductory Course*, New York, USA: Springer-Verlag, 1979.
[13] T. W. Hungerford, *Algebra*, Springer, (1974).
[14] S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra, 31 (2003), 4425–4443.
[15] S. P. Redmond, *Generalizations of the zero-divisor graph of a ring*, ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.) The University of Tennessee. (2001).
[16] E. Yetkin Celikel, *The triple zero graph of a commutative ring*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (in press).

(Received: December 14, 2016)
(Revised: June 12, 2017)

Ece Yetkin Celikel
Gaziantep University
Department of Mathematics
Faculty of Science and Art,
Gaziantep, Turkey
yetkinece@gmail.com